8. FROBENIUS GROUPS

§8.1. Permutation Representations

A **permutation representation** is one where the corresponding matrices are permutation matrices.

Theorem 1: If θ is the character of a permutation representation then gθ = #symbols fixed by g.

Proof: gθ is the trace of the corresponding matrix. This matrix has entries 0 and 1 only, and there is a 1 on the diagonal in those positions that correspond to a symbol fixed by g. ✋☺

A permutation group G is **transitive** permutation group on X if for all x, y ∈ X there exists α ∈ G such that xα = y. Let $\mathbf{G_{rs}}$ = {g | rg = s} and let $\mathbf{G_r} = G_{rr}$.
Then $G_r \le G$ for all r.

Theorem 2: If $G \le S_n$ is transitive then:

$$|G:G_r| = n \text{ for all } r.$$

Proof: If $g \in G_{rs}$ then $G_r g = G_{rs}$ so $G_{r1}, \ldots, G_{rn}$ are the left cosets of G_r in G. ✋☺

Theorem 3: If G is a transitive permutation group on *n* symbols, and θ is the permutation representation, then:

$$\sum_{g \in G} g\theta = |G|.$$

Proof: $\sum_{g\in G} g\theta = |\{(x, g) \mid xg = x\}|$

$$= \sum_{r=1}^{n} |G_r|$$

$$= n.\frac{|G|}{n} = |G|.$$ ✋☺

§8.2. Multiply Transitive Permutation Groups

A permutation group G, on a set X, is **T-transitive** if for all *t*-tuples $(x_1, \ldots, x_t)$ and $(y_1, \ldots, y_t) \in X^T$, with the x_i's distinct and the y_i's distinct, there exists $\alpha \in G$ such that for each i, $x_i\alpha = y_i$.

Examples 1: S_n is n-transitive, A_n is $(n-2)$-transitive.
$S_3 \times S_3$ is not transitive.
NOTE: Apart from S_n and A_n there are only four 4-transitive groups, called the Mathieu groups.

Theorem 4: If G is a 2-transitive permutation group and θ is the corresponding character then:

$$\sum_{g\in G} (g\theta)^2 = 2|G|.$$

Proof: G is transitive on $D = \{(x, y) \mid x \neq y\}$,
where $(x, y)g$ is defined to be (xg, yg).

Now the number of elements of D fixed by g is:

$$(g\theta))((g\theta) - 1).$$

So $\sum_{g\in G}[(g\theta)^2 - (g\theta)] = |G|$ and so:

$$\sum_{g\in G}(g\theta)^2 = \sum_{g\in G}(g\theta) + |G| = 2|G|.$$

Theorem 5:
Let G be a permutation group with character θ.
If G is transitive, χ_1 occurs in θ with multiplicity 1.
If G is doubly transitive, $\theta = \chi_1 + \chi_j$ for some non-trivial irreducible character χ_j.
Proof: If G is transitive, $\langle\chi_1 \mid \theta\rangle = \Sigma(g\theta)/|G| = 1$.
If G is doubly transitive, $\langle\theta \mid \theta\rangle = \Sigma(g\theta)^2/|G| = 2$.

§8.3. Frobenius Permutation Groups

Theorem 6: Suppose G is a transitive permutation group on n symbols, and only the identity fixes more than one symbol. Let $H = G_1$ and let χ_1 be the trivial character of G, θ the permutation character of G and let ψ be any irreducible character of H, of degree m. Then:

$$\psi^* = \psi^G - m(\theta - \chi_1)$$

is an irreducible character of G.
Proof: Let X = {g | g fixes exactly one symbol} and
Y = {g | g fixes no symbols}.
Then G = {1} + X + Y and each of X, Y is the disjoint union of conjugacy classes.

	1	X			Y		
θ	n	1	…	1	0	…	0
χ_1	1	1	…	1	1	…	1

Since $\langle \theta \mid \chi_1 \rangle > 0$, $\alpha = \theta - \chi_1$ is a character:

	1	X			Y		
θ	n	1	…	1	0	…	0
χ_1	1	1	…	1	1	…	1
α	$n - 1$	0	…	0	–1	…	–1

The elements of X consist of the non-trivial elements of the G_r, for $r = 1, 2, \ldots, n$.
Hence $|X| = n(h - 1)$ where $h = |H| = |G|/n$ and so

$$|Y| = n - 1.$$

Since the G_r are conjugate in G each conjugacy class within X contains an element of H.

Let $1 \neq x \in H$ and let $y \in C_G(x)$.
Then $1yx = 1xy = 1y$.
Since x fixes both 1 and 1y it must be that $1y = y$ and so:

$$y \in H.$$

Hence $C_G(x) = C_H(x)$ and so $|x^G| = n.|x^H|$.
If $g^{-1}xg = y \in H$, $1gy = 1xg = 1g$.
Clearly $y \neq 1$ and, since it fixes both 1 and 1g,

$$1g = 1 \text{ and so } g \in H.$$

It follows that $x^G \cap H = x^H$.

Hence the proportion of each conjugacy class within X that lies in H is $1/n$.

Let the conjugacy classes of H be $\Gamma_1, \ldots, \Gamma_k$ with sizes $h_1, \ldots, h_k$.
For each $r = 1, 2, \ldots, n$ choose $z_r \in G_{1r}$.
Then the conjugacy classes of G that lie within X are of the form:

$$\Omega_j = \sum_{r=1}^{n} z_r^{-1}\Gamma_j z_r \text{ for } j = 2, \ldots, n,$$

where Σ here denotes a disjoint union.
Moreover $\Omega_j \cap H = \Gamma_j$. So within X there are $k - 1$ conjugacy classes of sizes $nh_2, \ldots, nh_k$.

G	**1**	**X**			**Y**		
α	$n - 1$	0	…	0	–1	…	–1
ψ^G	mn	$\psi(\Gamma_2)$	…	$\psi(\Gamma_k)$	0	…	0
ψ^*	m	$\psi(\Gamma_2)$	…	$\psi(\Gamma_k)$	m	…	m
ψ	m	$\psi(\Gamma_2)$	…	$\psi(\Gamma_k)$	0	…	0

At this stage ψ^* is just a linear combination of irreducible characters, with integer coefficients. It may not be a character, let alone an irreducible one.

$$\langle\psi^*, \psi^*\rangle = \frac{m^2 + \sum_{j=2}^{n} nh_j|\psi(\Gamma_j)|^2 + (n-1)m^2}{nh}$$

$$= \frac{m^2n - m^2n + n\sum_{j=1}^{n} h_j|\psi(\Gamma_j)|^2}{nh} = \frac{nh}{nh} = 1.$$

Thus $\psi^G - m\alpha$ is an irreducible character or minus an irreducible character.
But $(\psi^G - m\alpha)(1) = mn - m(n - 1) = m > 0$ so $\psi^G - m\alpha$ is an irreducible character. ✋☺
Corollary: The set of permutations in G which fix no symbols, together with the identity, is a normal subgroup of G.
Proof: For all $x \in X \cap H$ there exists an irreducible character ψ of H such that $\psi(x) \neq \deg\psi$.
Hence x lies outside of the kernel of the representation corresponding to $\psi^G - (\deg\psi)\alpha$.
The intersection of these kernels must therefore be:

$$\{1\} + Y.$$

Example 2: $G = A_4$, $H \cong C_3$. X = (×××), Y = (××)(××), $h = 3$.

	I	**X**		**Y**
	I	**(×××)**	**(×××)**	**(××)(××)**
size	1	4	4	3
α	3	0	0	–1
ψ_1^G	4	1	1	0
$\psi_1^G - m\alpha$	1	1	1	1
ψ_2^G	4	ω	ω^2	0

$\psi_2^G - m\alpha$	1	ω	ω^2	1
ψ_3^G	1	ω^2	ω	1
$\psi_3^G - m\alpha$	1	ω^2	ω	1

§8.4. Abstract Frobenius Groups

A group G is a **Frobenius** group if there exists a proper non-trivial subgroup H of G such that $N_G(H) = H$ and $C_G(h) \leq H$ for all non-trivial $h \in H$.

Example 3: S_3 is a Frobenius group with H being any of the three subgroups of order 2.

Theorem 7: G is Frobenius if and only if there exists a proper non-trivial subgroup H of G such that $H \cap H^g = 1$ for all $g \in G - H$.
Proof: Suppose $H \cap H^g = 1$ for all $g \notin H$ and let $g \in C_G(h)$ for $1 \neq h \in H$.
Then $h \in H \cap H^g$ and so $g \in H$.

Suppose now G is Frobenius and let $|G:H| = N$.
Suppose $1 \neq h \in H \cap H^g$ for some $g \notin H$.
Let $1 = x_1$, $g = x_2$, x_3, … , x_N be a set of left coset representatives of H in G. Then H, H^g, H^{x_3}, …, H^{x_N} are the conjugates of H in G and, since $N_G(H) = H$, they are distinct.
Clearly $h^G = h^{Hx1} \cup h^{Hx1} ... \cup h^{Hx1}$.
Now $|hH^{x_i}| = |h^H|$ for each i.
Since $h \in h^{Hx_1} \cap h^{Hx_2}$, $|h^G| < N.|h^H|$.

But $C_G(h) = C_H(h)$ so $|h^G| = N.|h^H|$, a contradiction. ✋☺

Theorem 8: Suppose G is a Frobenius group. Then G is a transitive group of permutations on the left cosets of H and only the identity fixes more than one coset.
Proof: (Ha)g = Hag so G permutes the right cosets. Moreover it acts transitively.
If (Ha)g = Ha and (Hb)g = Hb for $g \neq 1$ then $g \in H^a \cap H^b$ $= (H \cap H^{ba^{-1}})^a$ so Ha = Hb. ✋☺
Theorem 9: A Frobenius group has a normal subgroup K and a subgroup H such that G = KH and $H \cap K = 1$. (G is called a split extension of K by H) ✋☺

The subgroup K is called the **kernel** of the Frobenius group.

Theorem 10: If A is any abelian group of odd order and G is A extended by a cyclic subgroup $H = \langle g \rangle$ of order 2 where g induces the automorphism $a \to a^{-1}$ on A, then G is a Frobenius group with kernel A.
Proof: Clearly A is a normal subgroup of G.
A typical element of G has the form a or ag where $a \in A$.
If a commutes with g then $ag = ga = a^{-1}g$ in which case $a^2 = 1$.
Since A has odd order we must have a = 1.
If ag commutes with g then $(ag)g = g(ag) = a^{-1}g^2$ in which case $a^2 = 1$ and so again a = 1.
Hence $H \cap H^a = 1$ and $H \cap H^{ag} = 1$ for all $a \in A$. ✋☺

Example 4: The dihedral group

$$D_{2n} = \langle A, B \mid A^n, B^2, BA = A^{-1}B\rangle$$

is a Frobenius group, with kernel $K = \langle A\rangle$. Any of the subgroups of order 2, for example $\langle B\rangle$, can play the role of H in the definition.

Theorem 11: If G is Frobenius with kernel K and complement H and $1 \neq k \in K$ then

$$C_G(k) = C_K(k).$$

Proof: Let $g \in C_G(k)$. If Hag = Ha then $aga^{-1} \in H$.
But $aka^{-1} \in C_G(aga^{-1})$ whence g = 1.
If g fixes no coset then, by the Frobenius theorem, $g \in K$.
✋☺

Theorem 12: Suppose G is Frobenius with kernel K and complement H.
Then the class equations for H and K have the forms:

$$|H| = N = 1 + h_2 + \ldots + h_s \text{ and}$$
$$|K| = M = 1 + k_2 * N + \ldots + k_t * N$$

and the class equation for G is:

$$|G| = 1 + k_2N + \ldots + k_tN + Mh_2 + \ldots + Mh_s.$$

Proof: Let $1 \neq x \in K$ and $1 \neq y \in H$.
Then $|x^G| = M|x^K|$ and $|y^G| = N.|y^H|$.
$|x^G|$ is a union of N conjugacy classes in K.
Every element $g \in G - K$ is conjugate to exactly one $h \in H$ so $|g^G| = M.|h^G|$. ✋☺

[Note that in the class equation given for G the terms are in non-descending order.]

Theorem 13 (THOMPSON): The kernel of a Frobenius group is nilpotent.
Proof: We omit the proof of this very deep theorem. ✋

So far in all our examples of Frobenius groups the kernel has been abelian. However it can be non-abelian.

Example 5: Let K be the group of all 3 × 3 uni-upper-triangular matrices over $\mathbb{Z}_7$. That is, the elements of K have the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$. This is a non-abelian group of order 343 = 7^3.
Extend K by H = ⟨g⟩ of order 3 such that:

$$g^{-1}\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} g = \begin{pmatrix} 1 & 2a & 4b \\ 0 & 1 & 2c \\ 0 & 0 & 1 \end{pmatrix}.$$

The resulting group G is a Frobenius group of order 1029 = 3.7^3 with Frobenius kernel K of order 343. Although K is not abelian, it is nilpotent, in accordance with Thompson's Theorem.

Theorem 14: Suppose the Frobenius group *G* has the class equation:

$$|G| = 1 + a_2 + \ldots + a_n \text{ where } a_2 \le a_3 \le \ldots \le a_n.$$

Let N = a_2, M = |G|/N and let a_r be the last term divisible by N.

Then :

(1) $M = 1 + a_2 + \ldots + a_r$;

(2) $N = \dfrac{a_{r+1} + \ldots a_{r+s}}{M} + 1$;

(3) $N \mid M - 1$;

(4) $N \mid a_i$ if $i \le r$;

(5) $M \mid a_i$ if $i > r$;

(6) if N is even, $a_i = N$ for $i \le r$.

Proof: Since N doesn't divide any Mh_i, $r = t$ and

$$a_i = \begin{cases} k_i N \text{ if } i \le r \\ M h_{i-t+1} \text{ if } i > r \end{cases}.$$

Parts (1) – (5) now follow.

(6) Let $h \in H$ have order 2. If $k \in K$ then:

$$k^{-1}kk^hk = k^hk = h^{-1}kk^hh \text{ so}$$

$kh^{-1} \in C_G(kk^h) \le K$ or $kk^h = 1$.

The first is a contradiction so $k^h = k^{-1}$ for all $k \in K$.

If $a, b \in K$, $(ab)^h = a^hb^h$ whence $(ab)^{-1} = a^{-1}b^{-1}$ and so $ab = ba$. Hence K is abelian. ✋☺

Theorem 15 (FROBENIUS TEST): Let p be prime. Suppose $|G| = pN$ and G has $p - 1$ conjugacy classes of size N. Then G is a Frobenius group with kernel G′ of order N.

Proof: Let h belong to a conjugacy classes of size N and let $H = C_G(h)$.

By the proof of the pN Test, each conjugacy class of size N contains exactly one non-trivial element of H.

Suppose $1 \neq h \in H \cap x^{-1}Hx$.
Then $H \leq C_G(h)$ and since they both have order r,
$H = C_G(h)$.
If $x \notin H$ then $h \neq xhx^{-1} \in H$.

So the conjugacy class containing *h* contains two elements of H, contradicting the *p*N Test.
Hence if $x \in G - H$, $H \cap x^{-1}Hx = 1$.
Thus G is a Frobenius group.
If the Frobenius kernel is K then $|K| = N$.

Now $G/K \cong H$ and so is abelian. Thus $G' \leq K$.
But, if x, y belong to the same conjugacy class of size N, then $x^{-1}y \in G'$ and so $|G'| \geq N$. Hence the Frobenius kernel is G′. 🖐☺

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